Wavelength Assignment to Minimize Requirement on Tunable Range of Optical Transceivers in WDM Networks

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Abstract-In **this paper we consider WDM networks with a tunable transmitter and a fixed-wavelength receiver at each station (similar results hold when the transmitter is fixed and the receiver is tunable). Traditionally, each station is required to be able to access all wavelength channels used in the network. Such requirement limits the number of wavelengths that can be exploited in a WDM network up to the size of the resolvable wavelength set of optical transceivers, which is very limited with current technology. In this paper we observe that this requirement is actually an overkill. To realize a communication topology, physical or logical, it is sufficient that the tunable range of the transmitter at each station covers all the wavelengths of the receivers at its neighboring stations. This observation leads to the study of optimal wavelength assignment to minimize the tunability requirement while still guaranteeing that each receiver has a unique wavelength channel. This optimization problem is shown to NP-complete in general and approximation algorithms with provable performance guarantees are presented. When the communication topologies are complete graphs, de Bruijin digraphs, Kautz digraphs, shuffle or rings, the optimal solutions are provided. Finally, we present tight lower bounds when the communication topology is a hypercube.**

Keywords: WDM, free spectral range, tunable range, minimum bandwidth, wavelength assignment.

I. Introduction

In WDM optical networks, either passive optical networks or wavelength routed optical networks [9], the multi-wavelength transmission/reception at each station is realized through either an array of fixed-wavelength transceivers or a tunable transceiver. A transceiver array has the advantage of being able to select the wavelength for each transceiver independently. However it becomes very bulky as the number of transceivers increases. On the other hand tunable transceiver has very limited tunable range with the current technology. The tunability of tunable transceivers is usually realized via tunable optical filters. Many tunable optical filters have a very limited resolvable wavelength set. These resolvable wavelength set in general consists of a number of contiguous wavelength channels, for an example, determined by the free spectral range. Traditionally, each station is required to be able to access all wavelength channels used in the network. Under such requirement, the set of working channels of the network is limited to the resolvable wavelength set of tunable transceivers. To increase the transmission capacity of the network, the transceivers at each station must be upgraded to support more resolvable wavelength set. Unfortunately, this is often not practical most of the time.

At the first glance, one might feel pessimistic on the transmission capacity of the WDM networks making use of tunable transceivers with limited resolvable wavelength channels. However, careful re-examining the operation principles of WDM networks reveals more optimistic discoveries. The requirement on stations to be able to access all working wavelength channels is actually an overkill in most situations. For an example, in many real applications, each station only communicates with a small number of stations among a potentially large population of stations in the entire network. To support the communications with these subset of stations, the tunable transceiver only have to access the wavelength channels used by these subset of stations. If the number of these channels are very small, then a tunable transceiver with a small resolvable wavelength set is sufficient to carry out all communications.

Based on this observation, one might be interested in finding out the minimum requirement on the tunable range of optical transceivers to support a communication topology, either physical or logical. This paper is intended to address this question. Each station is assumed to have a tunable transmitter and a fixed-wavelength receiver. The results can be extended to the opposite configuration in which the transmitter is fixed while the receiver is tunable. Each tunable transmitter can access the same number of

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contiguous resolvable wavelength channels while the resolvable wavelength set of different transmitters might be different. Such assumption reflects the free spectral range of many tunable filters [2]. These filters can operate on any contiguous resolvable wavelength channels as long as the number of these channels does not exceed the size of the free spectral range. The contiguous resolvable wavelength channels of any transmitter is referred to as a *waveband*. Then to support a given communication topology, the wavebands of the tunable transmitters and the wavelength channels of the fixed-wavelength receivers must be carefully selected such that for any link in the communication topology, the wavelength of the receiver at the destination station is within the waveband of the transmitter at the source destination. In addition, we require that each receiver owns a unique wavelength channel so as to maximize the number of working wavelength channels of the entire network. (Note that it's the set of wavelength channels used by the fixed-wavelength receivers, rather than the set of wavelength channels covered by the wavebands of the transmitters, that determines the working wavelength channels of the entire network.) Any waveband/wavelength assignment to the transmitters and receivers satisfying these conditions is said to be valid. The tunability requirement of any valid waveband/wavelength assignment is then simply one plus the maximum difference of the wavelength channels to any pair of receivers that can potentially talk to the same transmitter. Our objective is then to find a valid waveband/wavelength assignment with minimum tunability requirement.

The remaining of this chapter is arranged as follows. Section II formulate the minimum tunability problem into a graph-theoretic optimization problem. Section III studies the computational complexity of this problem in general communication topologies and provides approximation algorithms with provable performance. Section IV present the optimal wavelength assignment in complete graphs, de Bruijin/Kautz/shuffle digraphs, and rings respectfully, and tight lower bounds on minimum tunability in hypercubes. Finally Section V summarizes this paper.

II. Graph-Theoretic Formulation

The given communication topology in a WDM network is represented by a graph $G = (V, E)$ where $V = \{0, 1, \ldots, N-1\}$. Depending on whether the communications are bidirectional or unidirectional, G is expressed as an undirected graph or directed graph correspondingly. The wavelength channels are indexed by nonnegative integers. In the following, we prove that there always exists an optimal valid waveband/wavelength assignment in which the set of wavelength channels assigned to the receivers is exactly $\{0, 1, \cdots, N-1\}$.

Lemma 1: For any given communication topology with N stations, there always exists an optimal valid waveband/wavelength assignment in which the set of wavelength channels assigned to the receivers is exactly $\{0, 1, \ldots, N-1\}$.

Proof: Consider any optimal valid waveband/wavelength assignment in which the N wavelengths assigned to the N receivers are $w_0 < w_1 < \ldots < w_{N-1}$. If $w_0 \neq 0$, then replacing each wavelength w by $w - w_0$ results in another valid waveband/wavelength with the same tunability requirement. Let $0 = w'_0 < w'_1 < \ldots < w'_{N-1}$ denote the N wavelengths assigned to the N receivers in this new optimal waveband/wavelength assignment. If it is still not the desired, then choose the minimum k such that $w'_k > k$ and then replace each wavelength $w \geq w'_k$ by $w - w'_k + k$. The resulting waveband/wavelength assignment is still an optimal valid one. Such procedure can be repeated until a desired waveband/wavelength assignment is obtained. ■

From Lemma 1 we can restrict our attention to those valid wavebandlwavelength assignments in which he set of wavelength channels assigned to the receivers is exactly $\{0, 1, \ldots, N-1\}$. Furthermore, any such wavelength assignment to the receivers only can be extended to one or more valid waveband/wavelength assignment by assigning each transmitter a waveband containing the wavelengths of all receivers that it communicates with. Thus we can focus on only the wavelength assignment to the receivers with wavelengths $\{0, 1, \ldots, N-1\}$. As each receiver must have a unique wavelength, any wavelength assignment corresponds to a permutation over the set $\{0, 1, \cdots, N-1\}$. Thus the problem can be formulated as follows.

Minimum Tunability Problem Given a graph $G = (V, E)$, find

$$\Phi(G) = 1 + \min_{\pi \in P_V} \max_{v \in V} \left\{ \max_{(v,u) \in E} \pi(u) - \min_{(v,u) \in E} \pi(u) \right\}$$

where P_V is the set of all permutations on V.

A remark on the above description is that when G is a directed graph, the (v, u) represents the link from node v to node u and thus is an ordered pair. $\Phi(G)$ is exactly the minimum tunability.

An opposite of the minimum tunability problem is the **Maximum Concurrence Problem** in which the tunability w of the transmitters is given and we would like to assign as many wavelengths as possible to the receivers under the constraint that the wavelengths assigned all receivers that communicate with a common transmitter must within a waveband of length w. For any graph $G = (V, E)$ and the tunability $1 \leq w \leq |V|$, the maximum concurrence can be represented by

$$\Lambda_w(G) = \max\{|\{\lambda(v) \mid v \in V\}| : \max_{v \in V}\left\{\max_{(v,u)\in E}\lambda(u) - \min_{(v,u)\in E}\lambda(u)\right\} < w\}.$$

To calculate Φ (G), one may first calculate Λ_w (G) for each $1 \leq w \leq |V|$ and then obtain Φ (G) according to the following relation

$$\Phi(G) = \min\{1 \leq w \leq |V| : \Lambda_w(G) = |V|\}.$$

This approach will be used later in this paper.

III. Optimal Wavelength Assignment in Arbitrary Undirected Graphs

In this section, we first prove that the minimum tunability problem is NP-hard in general. A reduction will be made from the well-known *minimum bandwidth problem [6]*:

Minimum Bandwidth Problem Given a graph G = (V, E), find

$$BW(G) = 1 + \min_{\pi \in P_V}\max_{(u,v)\in E}|\pi(u) - \pi(v)|$$

where P_V is the set of all permutations on V.

Theorem 2: The minimum tunability problem is NP-Hard. It is even NP-Hard to approximate it within absolute error $N^{1-\epsilon}$ for any $\epsilon > 0$.

Proof: We reduce the problem of minimum bandwidth of cobipartite graph to the minimum tunability problem [6]. Let G = (U, V, E) be any cobipartite graph. Then for any $\epsilon \geq 0$, it is NP-hard to approximate BW (G) within absolute error of $(|U| + |V|)^{1-\epsilon}$ [6]. Without loss of generality, we assume that $|U| \leq |V|$. We construct graph $H = (V(H), E(H))$ as follows:

$$V(H) = \{a, b\} \cup U \cup U' \cup V,$$

where $U' = \{u' : u \in U\}$ is a copy of U.

$$E(H) = \{(a, u) : u \in U\} \cup \{(v, b) : v \in V\} \cup \{(u, u') : u \in U\} \cup \{(u', v) : (u, v) \in E(G)\}.$$

Let $K = (V(K), E(K))$ be the graph in which

$$V(K) = \{a, b\} \cup U',$$

and

$$E(K) = \{(a, u') : u \in U\} \cup \{(u', b) : u \in U\} \cup \{(u_1', u_2') : u_1, u_2 \in U; \exists v \in V, (u_1, v), (u_2, v) \in E(G)\}.$$

Then $\Phi(H) = \max\{BW(G), BW(K)\}$. Obviously,

$$BW(K) \leq 1 + |U'| = 1 + |U| \leq 1 + |V|$$

and

$$BW(G) \geq |V|.$$

Thus $BW(G) \leq \Phi(H) \leq 1 + BW(G)$. So the minimum tunability problem is at least as hard as the minimum bandwidth problem over cobipartite graphs. Therefore,the lemma follows from the NP-hardness and approximality of the minimum bandwidth problem over cobipartite graphs. ∎

In the next, we seek approximation algorithms for the minimum tunability problem in general graphs. The approximation algorithms can be obtained by reducing the minimum tunability problem to minimum bandwidth problem. For any graph G = (V, E), let $\overline{G} = (V, \overline{E})$ be the graph in which

$$\overline{E} = \{(v_1, v_2) : v_1, v_2 \in U; \exists v \in V, (v_1, v), (v_2, v) \in E\}.$$

It's obvious that $\Phi(G) = BW(\overline{G})$. Thus any approximation algorithms for minimum bandwidth problem can be applied to $\overline{G}$ to obtain an approximation algorithm for the minimum tunability of G. There are many approximation results on the minimum bandwidth problem that can be readily adopted for the minimum tunability problem. For examples, if the nodal degree is $\Theta(|V|)$, the minimum bandwidth problem is approximable within 3 [7]. If G is a caterpillar, the minimum bandwidth problem is approximable within O(log $|V|$) [3]. If G is asteroidal triple-free, the minimum bandwidth problem is approximable within 2 [8]. However, there are also some inapproximality results. For any $\varepsilon > 0$, the minimum bandwidth problem is not approximable within $1.5 - \varepsilon$ [1], and not approximable with an absolute error guarantee of $|V|^{1-\varepsilon}$ [6]. Even if G is a tree, it is still not approximable within $1.332 - \varepsilon$ for any $\varepsilon > 0$ [1].

Fortunately, most applications deals with special communication topologies, which allow for polynomial-time optimal solution. Before we move on to the these special communications topologies, we first give the following straightforward bounds on $\Lambda_w(G)$: for any graph $G = (V, E)$ and any $1 \leq w \leq |V|$, $w \leq \Lambda_w(G) \leq |V|$.

IV. Optimal Wavelength Assignment in Regular Graphs

A. Complete Graphs

The complete graph corresponds to a single-hop network if it represents the virtual topology embedded into the physical networks, or a all-to-all personalized communication request if it represents a communication pattern. We use K_N to denote the complete graph of N vertices. We show that for any $1 \leq$ w $\leq$ N, $\Lambda_w(K_N)$ = w. Obviously, A, $(K_N) \geq$ w. On the other hand, consider any feasible wavelength assignment $\lambda(\cdot)$. Let station a be the one assigned with the smallest wavelength $\lambda(a)$, and station b the one assigned with the largest wavelength X(b). As a and b are both neighbors of any other node other than a and b, $X(b) - \lambda(a) <$ w. Thus $\Lambda_w(K_N) \leq$ w. Therefore $\Lambda_w(K_N)$ = w, and therefore $\Phi(K_N)$ = N. Thus any wavelength assignment which uses w wavelengths is optimal. To make the wavelengths sharing the same workload, we should equally partition the stations into w subsets, and assign a distinct wavelength to all stations in the same subset.

*B. de Bruijin **Digraphs**, Kautz Digraphs and **Shuffles***

A de Bruijin digraph [4], [10] (or Kautz digraph [5], shuffle respectively) of size N and degree p is denoted by $D(N,p)$ (or $K(N,p)$, $S(N,p)$ respectively). We assume that in these graphs, the degree p is always selected to be a factor of the size N. For each $0 \leq i < \frac{N}{p}$, let $\mathcal{R}_i$ = $\left\{) \sqrt{} + \| \mid / \leq \| < \right\}$. Then $\mathcal{R}_i$ consists of exactly the immediate successors of some node in $D(N,p)$, $K(N,p)$ or $S(N,p)$, and thus can be assigned with at most min {p, w} wavelengths. Furthermore, in any optimal wavelength assignment the two sets of wavelengths assigned to $\mathcal{R}_i$ and $\mathcal{R}_j$ should be disjoint for any $0 \leq i < j < \frac{N}{p}$. So the total number of wavelength used is at most $\frac{N}{p}$ min {p, w} = min $\left\{\text{N}, \text{w}\frac{N}{p}\right\}$. On the other hand, there are many wavelength assignments which use min {N, $w\frac{N}{p}\}$ wavelengths. For an example, for each $0 \leq i < \frac{N}{p}$, we partition $\mathcal{R}_i$ into min {p, w} groups as equally as possible and then any min {p, w} contiguous wavelengths are assigned to these min $\{p$, w$\}$ groups. Therefore, A, $(D(n,p))$ = $\Lambda_w(K(n,p))$ = $\Lambda_w(S(n,p))$ = $\min\left\{N, w\frac{N}{p}\right\}$. This implies $\Phi(D(n,p)) = \Phi(K(n,p)) = \Phi(S(n,p)) = p$. Thus even with fixed-wavelength transmitters, certain degree of concurrence ($\frac{N}{p}$) is achievable. In particular, if the nodal degree is two, we can still achieve half of the full concurrence.

Consequently, as long as the number of resolvable wavelengths of each transmitter is no less than the degree of these graphs, full concurrence can be achieved. In particular, if the degree is equal to two, then any tunable transmitters are sufficient to achieve full concurrence. In general, when choosing the degree of these graphs as virtual topologies to be embedded into any given physical networks, the degree should be chosen to be no more than the number of resolvable wavelengths of the transmitters in order to achieve the full concurrence.

C. Rings

We begin with unidirectional rings. Let UR_N denote the unidirectional ring of N vertices $\{0, 1, \ldots, N-1)$. The wavelength assignment which assigns each vertex a distinct wavelength is feasible as each vertex has only one immediate successor. Thus for any $1 \leq$ w $\leq$ N, A, (UR_N) = N, and thereby $\Phi(UR_N)$ = 1. This implies that there is no need to use tunable transmitter at all in unidirectional rings. The fixed-wavelength transmitter is sufficient.

Now we consider bidirectional rings. Let BR_N denote the bidirectional ring of N vertices $\{0, 1, \ldots, N-1\}$. Without loss of generality, we assume that the vertices are numbered in the clockwise order. The wavelength routing in bidirectional rings will be reduced to the following optimal ring labeling problem in the bidirectional rings.
Optimal ring labeling problem: Given a ring of size N and an integer 0 < w < N, assign a label C(i) to each $0 \leq i <$ N such that

$$|\ell(i) - \ell((i+1) \bmod N)| < w$$

and $|\{\ell(i) : 0 \leq i < N\}|$ is maximized. The maxima is denoted by Γ_w(N).

Then for any $1 \leq$ w $\leq$ N,

$$\Lambda_w(BR_N) = \begin{cases} 2\Gamma_w\left(\frac{N}{2}\right) & \text{if N is even,} \\ \Gamma_w(N) & \text{if N is odd.} \end{cases}$$

In fact, when N is even, let BR'_N and BR''_N be the two rings consisting of all even nodes and all odd nodes in BR_N respectively. Thus a wavelength assignment in BR_N is feasible if and only if it induces a feasible ring labeling with respect to w in both BR'_N and BR''_N. Thus, to maximize the number of wavelength used, the wavelengths/labels assigned to these two small rings should be disjoint. Hence A, (BR_N) is twice of $\Gamma_w\left(\frac{N}{2}\right)$. Now we assume that N is odd. Consider the ring G in which nodes are arranged clockwise in the following order

$$0, 2, 4, \cdots, N-3, N-1, 1, 3, 5, \cdots, N-4, N-2.$$

It can be shown that a wavelength assignment in BR_N is feasible if and only if it induces a feasible ring labeling with respect to w in G. Thus $\Lambda_w(BR_N) = \Gamma_w(N)$.

In the next we find the optimal ring labeling. If w = 1, then all nodes in the ring must have the same labeling and thus Γ_1 (N) = 1. If w > 2 and N is even, we consider the following ring labeling $\ell(\cdot)$: f?(O) = 0, $\ell\left(\frac{N}{2}\right)$ = N $-$ $1, \ell(i) = 2i-1$ for any $0 < i < \frac{N}{2}, \ell(i) = 2(N-i)$ for any $\frac{N}{2} < i <$ N. If w > 2 and N is odd, we consider the following ring labeling $\ell(\cdot)$: $\ell(0)$ = 0, $\ell(i) = 2i-1$ for any $0 < i \le \lfloor \frac{N}{2} \rfloor, \ell(i) = 2(N-i)$ for any $\lceil \frac{N}{2} \rceil \le i < N$. In both cases, $|\ell(i) - \ell$ ((i + 1) mod $N)| \le 2 \le$ w $-$ 1 and $|\{\ell(i) : 0 \le i < N\}|$ = N. Thus we also have Γ_w (N) = N if w > 2. Now we assume that w = 2. Then for any 0 < i < N, $|\ell(i) - \ell(0)| \le \min\{i,$ N $- \, i\} \le \lfloor \frac{N}{2} \rfloor$. Thus Γ_2 (N) $\le$ 1 + $\lfloor \frac{N}{2} \rfloor = \lceil \frac{N+1}{2} \rceil$. On the other hand, there is a ring labeling that uses $\lceil \frac{N+1}{2} \rceil$ different labels. If N is odd, we consider the following ring labeling $\ell(\cdot)$: $\ell(0) = 0, \ell(i) = \ell(N -$ i) = i for any $0 < i \le \lfloor \frac{N}{2} \rfloor$. If N is even, we consider the following ring labeling $\ell(\cdot)$: C(0) = $0, \ell\left(\frac{N}{2}\right) = \frac{N}{2}, \ell(i) = \ell(N-i) = i$ for any $0 < i < \frac{N}{2}$. Thus Γ_2 (N) = $\lceil \frac{N+1}{2} \rceil$ for any N. In summary, for any $N > 1$,

$$\Gamma_w(N) = \begin{cases} 1 & \text{if} w = 1, \\ \lceil \frac{N+1}{2} \rceil & \text{if} w = 2, \\ \text{N} & \text{if} w > 2. \end{cases}$$

Therefore,

$$\Lambda_w(BR_N) = \begin{cases} 2 - \text{N} \bmod 2 & \text{if} w = 1, \\ 2\lfloor \frac{N}{4} \rfloor + \left\lceil \frac{(N-1) \bmod 4}{4} \right\rceil + 1 & \text{if } w = 2, \\ \text{N} & \text{if } w > 2. \end{cases}$$

$$\Phi(BR_N) = \begin{cases} 2 & \text{if N = 2 or 4,} \\ 3 & \text{if } N \ge 5 \text{ or } N = 3. \end{cases}$$

This implies that to achieve the full concurrence the number of resolvable wavelengths can be as low as three. If each transmitter can tune to only two contiguous wavelengths, we can still achieve the full concurrence when N = 2 or 4, and around half of the full concurrence if N $\ge$ 5 or N = 3. If the transmitters are fixed, the concurrence is very poor.

D. Hypercubes

Let C_n to denote the n-dimensional hypercube. Without loss of generality, we always assume that the wavelengths are positive integers and the lowest wavelength is 1 which is assigned to node 0. Consider any optimal wavelength assignment in C_n which achieves full concurrence while requires minimum tunability. Note that there are $\binom{n}{2}$ nodes which have distance of two from node 0, and all of them have a unique larger wavelength. One of them must have wavelength at least $\binom{n}{2}$ + 1. Hence $\Phi(C_n) \ge 1 + \binom{n}{2}$. This lower bound is also sufficient when n $\le$ 4. If n = 2, 1 + $\binom{n}{2}$ = 2. The following feasible wavelength assignment achieves full concurrence, and the difference between the wavelengths assigned to two neighbors of any node is at most 2:

$$\text{X(00)} = 1, \lambda(11) = 2, \lambda(01) = 3, \lambda(10) = 4.$$

If n = 3, 1 + $\binom{n}{2}$ = 4. The following feasible wavelength assignment achieves full concurrence, and the difference between the wavelengths assigned to two neighbors of any node is at most 4:

$$\text{X(000)} = 1, \text{X(011)} = 2, \text{X(101)} = 3, \text{X(110)} = 4,$$
$$\text{X(001)} = 5, \text{X(010)} = 6, \text{X(107)} = 3, \text{X(111)} = 8.$$

If n = 4, 1 + $\binom{n}{2}$ = 7. The following feasible wavelength assignment achieves full concurrence, and the difference between the wavelengths assigned to two neighbors of any node is at most 7:

$\lambda(0000)$ = 1	X(0011) = 2
X(0101) = 3	X(0110) = 4
X(1001) = 5	X(1010) = 6
X(1100) = 7	X(1111) = 8
$\lambda(0001)$ = 9	$\lambda(0010)$ = 10
$\lambda(0100)$ = 11	X(0111) = 12
X(1000) = 13	X(1011) = 14
X(1101) = 15	X(1110) = 16

When n $\ge$ 5, we have $\Phi(C_n) \ge \binom{n}{2} + \left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor + 1$. Suppose to the contrary. Then all $\binom{n}{2}$ nodes which contain exactly two l's must have wavelengths of no more than $\binom{n}{2} + \left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor$. Let a be the node with the smallest wavelength among these $\binom{n}{2}$ nodes. Then the wavelength assigned to a is at most $\left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor$ + 1. Among the wavelengths assigned to those $\binom{n-2}{2}$ nodes which contain exactly four l's and have distance of two from a, the maximum is at least 1 + $\binom{n}{2}$ + $\binom{n-2}{2}$. Therefore,

$$\Phi(C_n) \ge 1 + \left(1 + \binom{n}{2} + \binom{n-2}{2}\right) - \left(\left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor + 1\right)$$
$$= \binom{n}{2} + \left\lceil \frac{\binom{n-2}{2}}{2} \right\rceil + 1$$

which is a contradiction.

In the next, we present another stronger lower bound on $\Phi(C_n)$ when n is large. We consider the wavelengths assigned to all 2^{n-1} nodes of even parity in any optimal wavelength assignment. Suppose that node a has the lowest wavelength and node b has the largest wavelength.

Let m be the Hamming distance between a and b. Let $a = v_0, v_1, \cdots, v_{\frac{m}{2}} = b$ be the nodes of even parity along a shortest path from a to b. Since the differences between the two wavelengths assigned to a = v_0 and $v_{\frac{m}{2}} = b$ is at least $2^{n-1} - 1$, there exist some $0 \leq i < \frac{m}{2}$ such that the difference of the two wavelengths assigned to v_i and v_{i+1} is at least $\left| \frac{2^{n-1}-1}{\frac{m}{2}} \right| = \left\lceil \frac{2^n-2}{m} \right\rceil$. So when n is even, $m \leq n$ and thus

$$\Phi(C_n) \geq \left\lceil \frac{2^n-2}{n} \right\rceil + 1;$$

when n is odd, $m \leq n - 1$ and thus

$$\Phi(C_n) \geq \left\lceil \frac{2^n-2}{n-1} \right\rceil + 1.$$

In summary, for and $2 \leq n \leq 4$,

$$\Phi(C_n) = \binom{n}{2} + 1.$$

for any $n \geq 5$, if n is even,

$$\Phi(C_n) \geq \max\left\{ \binom{n}{2} + \left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor, \left\lceil \frac{2^n-2}{n} \right\rceil \right\} + 1$$

and if n is odd,

$$\Phi(C_n) \geq \max\left\{ \binom{n}{2} + \left\lfloor \frac{\binom{n-2}{2}}{2} \right\rfloor, \left\lceil \frac{2^n-2}{n-1} \right\rceil \right\} + 1.$$

V. Conclusion

This paper studies the minimum tunability requirement of a WDM network with a tunable transmitter and a fixed-wavelength receiver at each station to achieve full transmission concurrence. The problem is proved to be NP-hard in general. In addition, it is even NP-Hard to approximate it within absolute error $N^{1-\epsilon}$ for any $\epsilon > 0$. However, polynomial time approximation algorithms can be obtained by reducing it to the well-studied minimum bandwidth problem. When the communication topologies are complete graphs, de Bruijin digraphs, Kautz digraphs, shuffle or rings, the optimal solutions are provided. Finally, we present tight lower bounds when the communication topology is a hypercube.

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